Problem 6

Verify that $y(x) = x/(x+1)$ is a solution to the differential equation

$$y + \frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{x^3 + 2x^2 - 3}{(1+x)^3}.$$

Solution

Take derivatives of the given function.

$$\begin{aligned}
y(x) &= \frac{x}{x+1} \\
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{1 \cdot (x+1) - (1) \cdot x}{(x+1)^2} = \frac{1}{(x+1)^2} \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{1}{(x+1)^2}\right] = -\frac{2}{(x+1)^3} \\
\frac{d^3y}{dx^3} &= \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left[-\frac{2}{(x+1)^3}\right] = \frac{6}{(x+1)^4}
\end{aligned}$$

Now plug these formulas into the ODE and check to see if the left side is equal to the right side.

$$\begin{aligned}
y + \frac{d^2y}{dx^2} &\stackrel{?}{=} \frac{dy}{dx} + \frac{x^3 + 2x^2 - 3}{(1+x)^3} \\
\frac{x}{x+1} - \frac{2}{(x+1)^3} &\stackrel{?}{=} \frac{1}{(x+1)^2} + \frac{x^3 + 2x^2 - 3}{(1+x)^3} \\
\frac{x(x+1)^2}{(x+1)^3} - \frac{2}{(x+1)^3} &\stackrel{?}{=} \frac{x+1}{(x+1)^3} + \frac{x^3 + 2x^2 - 3}{(x+1)^3} \\
\frac{x(x+1)^2 - 2}{(x+1)^3} &\stackrel{?}{=} \frac{x^3 + 2x^2 - 3 + (x+1)}{(x+1)^3} \\
\frac{x^3 + 2x^2 + x - 2}{(x+1)^3} &= \frac{x^3 + 2x^2 + x - 2}{(x+1)^3}
\end{aligned}$$

Since this is a true statement, $y(x) = x/(x+1)$ is a solution to the ODE.